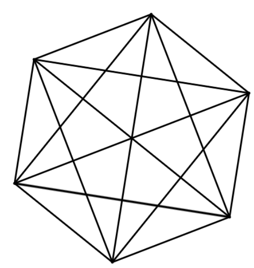
UCD Enrichment Programme in Mathematics
Selection Test 27 February 2016

1. I have two egg timers. The first can time an interval of exactly 7 minutes. The second can time an interval of exactly 9 minutes. Explain how I can use them to boil an egg for exactly 3 minutes?

2. Consider the hexagon shown below. Alternately, two players play the following game: in one move one player



selects one edge and colours it red and then the other player selects one remaining edge and colours it green. The winner is the player to complete first a triangle in their colour.

Show that the player who goes first has a strategy that will always guarantee a win in four moves.

3. (a) Show that the greatest common divisor of $(n+1)!+1$ and $n!+1$ is 1, for all integers $n \geq 1$.
   (b) For any $n > 1$, find integers x, y such that
$$((n+1)!+1)x + (n!+1)y = 1.$$
[Recall that $n! = 1 \times 2 \times \cdots \times n$ for any $n > 1$.]

4. Prove that for any positive real numbers a, b and c we have
$$\frac{2a+b}{b+2c} + \frac{2b+c}{c+2a} + \frac{2c+a}{a+2b} \geq 3.$$

5. ABC is an acute triangle and D is a point on the segment BC. Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ passing through B, D and C, D respectively intersect for the second time at P, where P lies inside of triangle ABC. Denote by R the intersection of $\mathcal{C}_1$ and AB and by Q the intersection of $\mathcal{C}_2$ and AC.

   Prove that P lies on the circumcircle of triangle QAR.

6. 25 boys and 25 girls are at a party. Each boy likes at least 13 girls, and each girl likes at least 13 boys.

   Show that there must be a boy and girl at the party who like each other.

7. On sides AB, BC and CA of triangle ABC we consider the points M, N and P respectively such that
$$\frac{AM}{MB} = \frac{BN}{NC} = \frac{CP}{PA} = \frac{1}{2}.$$
Prove that:
$$(a)\ [AMN] = \frac{1}{9}[ABC] \qquad (b)\ [MNP] = \frac{1}{3}[ABC].$$

8. Richard and nine other people are standing in a circle. All ten of them think of an integer (that may be negative) and whisper their number to both of their neighbours. Afterwards, they each state the average of the two numbers that were whispered in their ear.

   Richard states the number 10, his right neighbour states the number 9, the next person along the circle states the number 8, and so on, finishing with Richard's left neighbour who states the number 1.

   What number did Richard have in mind?

9. For each positive integer n let $s_n = n! + 20!$.

   (a) Let $q > 20$ be a prime number. Prove that there are only a finite number of positive integers k for which q divides s_k.

   (b) Find with proof all prime numbers p for which there exists a positive integer m such that p divides s_m and s_{m+1}.

10. For a real number x denote by $[x]$ the greatest integer not exceeding x.

    (a) Find with proof all positive integers k for which $[\sqrt[3]{k^3 + 20k}] \neq k$.

    (b) Prove that if n is a positive integer, then $\left[n + \sqrt{n} + \frac{1}{2}\right]$ is not the square of an integer.